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LETTER TO THE EDITOR

On a class of non-completely integrable equations with power-like nonlinearities and factorised associated linear operators

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Abstract. We explicitly build exponential-type bi-solitons of

$$L_qK = \sum_{i+j=1}^{i+j=q} a_{ij}\partial_{x^i t^j}^{i+j} K = \text{constant} \times K^{N-1}K_x \qquad N \text{ integer} \geqslant 2$$

or equivalently $L_qG = \text{constant}\ (G_x)^N$ for the potentials $G_x = K$. We assume both that their denominators have no soliton couplings and that L_q are either factorised linear operators or germs of factorised operators. K^N and $K^{N-1}K_x$ nonlinearities with associated factorised linear operators belong to a class of non-integrable equations sharing similar properties.

Due to the lack of methods to solve them, the non-integrable, nonlinear equations are actually less popular among physicists than the integrable ones. However, they are also interesting to study (Makhankov 1978). It appears that there exists a class of non-integrable equations in $1+1$ dimensions with the following features.

(i) The nonlinearities are of the power type K^N, $K^{N-1}K_x$ (K being the solution of the equation and N an integer equal to 2 or greater) and may be other ones, for instance K^2+KK_x belongs to that class.

(ii) Let us consider the linear part L_qK of the equation where L_q is a qth-order differential operator in $1+1$ dimensions. Then, either L_q is a factorised operator, or in the $K^{N-1}K_x$ case, L_N is a germ differential operator which becomes a factor of L_q when $q>N$.

(iii) Let us define bi-solitons as solutions with two variables $\omega_i = \exp(\gamma_i x + \rho_i t)$, $i=1,2$, such that there exist powers of the solutions which are rational functions. Then their denominators are functions of $\Delta = 1+\omega_1+\omega_2$ without the couplings terms constant $\omega_1\omega_2$, if we do not consider trivial bi-solitons.

(iv) There exists a direct method by which we simultaneously build both the linear operator L_q and the appropriate power of the solution.

We recall that these features were recently obtained for K^2 (Cornille and Gervois 1981, 1982a, b, c), $K^N N \geqslant 2$ (Cornille 1982), $K^2+\text{constant}\ KK_x$ and KK_x alone (Cornille and Gervois 1982a, b, c). In this last case, the term generating higher-order linear operators was the second-order linear differential operator of Burger's equations.

However, K^2K_x is also a classical nonlinearity and it seems worthwhile to know whether or not these above properties are particular to KK_x or valid for the more general $K^{N-1}K_x$ case. This is the aim of the present letter. Burger's equation is equivalent to a second-order linear differential equation and higher-order linear differential equations give well defined sums of nonlinearities which do not reduce to KK_x. However, as we shall see, the germs which for $N>2$ generalise the Burger's one, are still defined by a set of linear differential equations but are applied to *different powers of the same function.*

Here, we briefly report results on the class of nonlinear equations

$$L_qG = \sum_{i+j=1}^{i+j=q} a_{ij}\partial^{i+j}_{x^it^j}G = \text{constant}\times[G_x]^N \qquad N \text{ integer} \geq 2 \tag{1}$$

written down for the potentials G with $G_x = K$. Complete proofs and details will be provided elsewhere. We still assume both that the denominators of the bi-solitons have no soliton couplings ($\Delta = 1+\Sigma\omega_i$) and either L_q, $q>N$, is a factorised operator or $L_q = \tilde{l}_N$, $q=N$, is a germ of factorised operators (for $q>N$). As in our previous works we use a direct method building simultaneously L_q and the power N of G_x.

We summarise our general scheme in order to discover the bi-solitons.

Firstly we define a germ linear differential operator $\tilde{l}_N(x,t)$ of order N such that for any differentiable $F(\Delta)$ and any differentiable $\Delta(x,y)$ we have the property

$$[\tilde{l}_N(x,t) - \tilde{a}_{N0}(\Delta_x)^N\partial^N_{\Delta^N}]F(\Delta) \equiv 0 \tag{2}$$

which means that the coefficients of $\partial_\Delta F, \partial_{\Delta^2}F, \ldots, \partial^{N-1}_{\Delta^{N-1}}F$ are identically zero.

Secondly, for the concern of the present letter, we restrict Δ to

$$\Delta(x,t) = 1+\sum_1^2 \omega_i \qquad \omega_i = \exp(t+\gamma_i x) \qquad \gamma_1 \neq \gamma_2 \tag{3}$$

and make explicit the restrictions on the parameters of $\tilde{l}_N$ in order that equation (2) leaves two really different γ_i values.

Thirdly we introduce $l_{q-N}(t)$, a linear differential operator of order $q-N$ in $\partial^i_{t^i}$. From equation (3), we note that $\Delta_t = \Delta - 1$, $\partial_t = (\Delta-1)\partial_\Delta, \ldots$ and it follows that l_{q-N} can also be written down with the variables Δ, ∂_Δ.

Fourthly we define $L_q = \tilde{l}_N l_{q-N}$ and assume $G \equiv G(\Delta)$, with Δ as written down in equation (3). From equation (2) we find

$$L_qG = \tilde{a}_{NO}(\Delta_x)^N\partial^N_{\Delta^N}(l_{q-N}G)$$

and further we assume

$$\partial^N_{\Delta^N}l_{q-N}(t)G(\Delta) \equiv \nu(\partial_\Delta G(\Delta))^N. \tag{4}$$

It follows from equations (2)–(4) that the factorised L_q is associated to equation (1)

$$L_qG = l_{q-N}\tilde{l}_NG = \nu\tilde{a}_{NO}(\Delta_x)^N(G_\Delta)^N = \nu\tilde{a}_{NO}(G_x)^N. \tag{1'}$$

In the following we begin with the determination of germs $\tilde{l}_N$ satisfying equations (2), (3). Later on we solve the ordinary differential equation (4). We build, step by step, the operator $l_{q-n}(t)$ or $l_{q-n}(\Delta)$ in such a way that $\partial^N_{\Delta^N}(l_{q-N})$ when applied to the solution $G(\Delta)$ reproduces exactly $(G_\Delta)^N$.

Germs $\tilde{l}_N$ *and arbitrary* $\Delta(x,y)$. We start with

$$\tilde{l}_N = \sum_{i+j=1}^{N} \tilde{a}_{ij}\partial^{i+j}_{x^it^j}.$$

In equation (2), the coefficient of $\partial_{\Delta^N}^N F$ being only $(\Delta_x)^N$, it follows for $\tilde{l}_N$ that $\tilde{a}_{ij}=0$ for $i+j=N$, except $\tilde{a}_{NO}$ and we can write

$$\tilde{l}_N=\sum_{i+j=1}^{N-1}\tilde{a}_{ij}\partial_{x^it^j}^{i+j}+\tilde{a}_{NO}\partial_{x^N}^N.$$

Equation (2) must be valid for any $F(\Delta)$ and consequently for $F=\Delta^i$ we necessarily have

$$\tilde{l}_N\Delta^i=0 \qquad i=1,2,\ldots,N-1. \tag{5}$$

That this necessary condition is sufficient can be verified. For $N=2$, equation (2) leads to $(\tilde{l}_2\Delta)G_\Delta\equiv 0$, for arbitrary G we have $\tilde{l}_2\Delta=0$ and we recover the linear Burger's condition. For $N=3$, equation (2) gives

$$(\tilde{l}_3\Delta)G_\Delta+(\tfrac{1}{2}\tilde{l}_3\Delta^2-\Delta\tilde{l}_3\Delta)G_{\Delta\Delta}\equiv 0$$

and consequently $\tilde{l}_3\Delta=\tilde{l}_3\Delta^2=0$. We find from equation (2) and $N=4$ that $(\tilde{l}_4\Delta)G+(\frac{1}{2}\tilde{l}_4\Delta^2-\Delta\tilde{l}_4\Delta)G_{\Delta\Delta}+(\frac{1}{6}\tilde{l}_4\Delta^3-\frac{1}{2}\Delta\tilde{l}_4\Delta^2+\Delta^2\tilde{l}_4\Delta)G_{\Delta\Delta\Delta}\equiv 0$, consequently $\tilde{l}_4\Delta=\tilde{l}_4\Delta^2=\tilde{l}_4\Delta^3=0$. Equation (5) represents implicit constraints on both $\tilde{l}_N$ and Δ and for a very simple Δ we give the explicit relations.

Germs $\tilde{l}_N$ and $\Delta=1+\Sigma_1^2\exp(\gamma_i x+t)$. For $N=2$ and an arbitrary operator $\tilde{l}_2=\partial_t+a_{10}\partial_x+a_{20}\partial_{x^2}^2$, (equation (5)) leads to $\Delta_t+a_{10}\Delta_x+a_{20}\Delta_{xx}=0$ or $1+a_{10}\gamma_i+a_{20}\gamma_i^2$. We have a Burger's family of $\tilde{l}_2$ operators with two arbitrary parameters a_{10}, a_{20}, $a_{10}^2\neq 4a_{20}$.

For simplicity, in the $N=3$ case we put $a_{02}=0$ so that we start with an arbitrary four-parameter operator $\tilde{l}_3=\partial_t+\tilde{a}_{11}\partial_{xt}^2+\tilde{a}_{10}\partial_x+\tilde{a}_{20}\partial_{x^2}^2+\tilde{a}_{30}\partial_{x^3}^3$ and require equation (5). We find two linear differential conditions on the Δ: $\Delta_t+a_{10}\Delta_x-2a_{30}\Delta_{xxx}=0$, $a_{11}\Delta_t+a_{20}\Delta_x+3a_{30}\Delta_{xx}=0$. An elementary calculation shows that we have two arbitrary parameters with

$$\frac{3}{2}+\frac{\tilde{a}_{20}^2}{3\tilde{a}_{30}}\gamma_i+\gamma_i^2\tilde{a}_{20}=0 \qquad \tilde{a}_{20}^3\neq 54\tilde{a}_{30}^2$$

$$\tilde{l}_3=\partial_t+\frac{9\tilde{a}_{30}}{2\tilde{a}_{20}}\partial_{xt}+\left(\frac{2\tilde{a}_{20}^2}{9\tilde{a}_{30}}-\frac{3\tilde{a}_{30}}{\tilde{a}_{20}}\right)\partial_x+\tilde{a}_{20}\partial_{x^2}^2+\tilde{a}_{30}\partial_{x^3}^3.$$

For simplicity in the discussion we have found it more convenient to restrict the general starting $\tilde{l}_N$ to

$$\tilde{l}_N(x,t)=\sum_1^N\tilde{a}_{i0}\partial_{x^i}^i+\sum_{i=0}^{N-2}\tilde{a}_{i1}\partial_{x^it}^{i+1}+\sum_{i=0}^{N=4}\tilde{a}_{i2}\partial_{x^it^2}^{i+2}+\ldots$$

with the $\tilde{a}_{ij}$ constrained by equations (3)–(5). For $\tilde{l}_4=\partial_t+\tilde{a}_{02}\partial_{t^2}^2+\tilde{a}_{11}\partial_{xt}^2+\tilde{a}_{21}\partial_{x^2t}^3+\tilde{a}_{10}\partial_x+\tilde{a}_{20}\partial_{x^2}^2+\tilde{a}_{30}\partial_{x^3}^3+\tilde{a}_{40}\partial_{x^4}^4$ we start with seven parameters which must be such that $\tilde{l}_4\Delta=\tilde{l}_4\Delta^2=\tilde{l}_4\Delta^3=0$ and Δ given by equation (3). The analysis is more tedious but we still find a two arbitrary parameter family of $\tilde{l}_4$. In the general $\tilde{l}_N$ case, a counting argument of the number of $\tilde{a}_{ij}$ compared with the number of constraints such that really two independent γ_i values survive shows that we must have, at the end, two arbitrary parameters.

Compatible G and l_{q-N} for simple examples. Our aim is to build simultaneously G and l_{q-N} satisfying equation (4).

(i) If l_{q-N} is the identity, $L_q\equiv\tilde{l}_N$ and (see table 1) we have the solution $G=$ constant $\log\Delta$ which for $N=2$ reduces to the Burger's equation solution.

Table 1. Simple solutions.

$L_qG = \tilde{l}_N l_{q-N} G = \tilde{a}_{NO}\nu(G_x)^N$		$\Delta = 1 + \sum_1^2 \exp(t+\gamma_i x)$	
$\tilde{l}_N G(\Delta) = \tilde{a}_{NO}(\Delta_x)^N G_{\Delta^N}$		$\partial_{\Delta^N}^N(l_{q-N}(t))G(\Delta) = \nu(G_\Delta)^N$	
$G = \log\Delta$	$l_{q-N} = \text{Identity}$	$\nu = (-1)^{N+1}(N-1)!$	$q = N \geqslant 2$
$G = \Delta^{[(q-N)/(N-1)]}$	$l_{q-N} = \prod_{s=(q-N)/(N-1)}^{[N(q-N)/(N-1)]-1} \left(1+\frac{\partial_t}{s}\right)$	$\nu = \frac{\prod_{s=0}^{N-1} N(q-N)/(N-1)+s}{[(q-N)/(N-1)]^N}$	$q \geqslant N+1$

(ii) Let us try to find $G = \text{constant}\ \Delta^{-p}$. The RHS of equation (4) is proportional to $\Delta^{-(N+1)p}$ and l_{q-N} is an operator such that

$$l_{q-N}\Delta^{-p} = \prod_{s=p}^{Np-1}\left(1+\frac{\partial_t}{s}\right)\Delta^{-p} = \Delta^{-Np}.$$

This operator has $(N-1)p = q-N$ terms (see table 1).

(iii) Solutions mixing $\log\Delta$ and Δ^{-1} powers. The general solution can be written $G = a\log\Delta + \Sigma_1^k a_i\Delta^{-i}$ where $k = (q-N)(N-1)^{-1}$ is an integer, $q = 2N-1$, $3N-2$, $4N-3, \ldots$ and $l_{q-N} = \Sigma_{j=1}^{q-N}\eta_{j-1}\partial_{t^j}$. We must find the compatible sets of ν, a, (a_i), (η_j) values so that equation (4) is fulfilled. We rewrite l_{q-N} in terms of Δ variables instead of $t: \partial_{t^m} = \Sigma_{l=1}^m(\Delta-1)^l\mathscr{C}_j^l\partial_{\Delta^l}^l$ and the Stirling numbers $\mathscr{C}_j^l$ (Abramowitz and Stegun 1964) appear, then we replace the set (η_l) by $(\bar{\eta}_l)$

$$\bar{\eta}_0 = \eta_0 \qquad \bar{\eta}_l = \sum_{j=l}^{q-N-1}\mathscr{C}_j^l\eta_j \tag{6}$$

and the $\bar{\eta}_l$ (or η_l) are recursively determined by triangular relations in terms of ν, a, (a_i). We eliminate ν, (η_l) and we find that a, (a_i) satisfy algebraic equations. This solution is too complicated to be explicitly determined in the general case and we sketch only two simple cases. First, there is a general solution $G = -\log\Delta + \Sigma_1^k i^{-1}\Delta^{-i}$ and the set of triangular relations for the $\bar{\eta}_l$ are given in table 2 in terms of q, N and a coefficient $\mu(j,q,N)$ defined by the identity:

$$\Delta^{N+(q-1)/(N-1)}\left(\sum_1^{(q-1)(N-1)^{-1}}\Delta^{-m}\right)^N = \sum_{j=-(q-1)(N-1)^{-1}}^{q-N-1}\mu(j,q,N)^{\Delta-j}. \tag{7}$$

Secondly we consider $G = a\log\Delta + \Delta^{-1}$, l_{q-N} is an operator of order $N-1$ and $q = 2N-1$. The set of triangular relations for the $(\bar{\eta}_l)$ contain a simple one $(a+1)(\Sigma\, l!\,\bar{\eta}_l(-1)^l) = 0$. $a = -1$ is one of the previous solutions $G = -\log\Delta + \Delta^{-1}$ and we look at the other way $a+1 \neq 0$. From the other relations (table 2), we can eliminate $(\bar{\eta}_l)$, ν and obtain an algebraic equation for a. We give the explicit results for $N = 3$, $q = 5$ and $N = 4$, $q = 7$.

$G = \Delta^{-p}(b+\Delta^{-1})$. We want to find compatible b, p values so that both G and l_{q-N} satisfy equation (4). We integrate, with respect to Δ, both sides of equation (4) N times

$$l_{q-N}G = X(\Delta^{-1}) \qquad X = \sum_{M=0}^{N}\nu\Delta^{-(M+Np)}(p+1)^M b^{N-M} c_N^M\left(\prod_{s=0}^{N-1}(pN+M+s)\right)^{-1} \tag{4'}$$

Table 2. Solutions with a log Δ term.

$G=-\log\Delta+\sum_{1}^{(q-N)(N-1)^{-1}} m^{-1}\Delta^{-m}$	$l_{q-N}=1+\sum_{j=1}^{q-N}\eta_{j-1}\partial_t^j$			
$\nu=(N-1)!$	$\bar\eta_l$ (equation (6)) $q=N+l(N-1)$	$l=1,2,\ldots$		
$\sum_{l=j}^{q-N-1}\bar\eta_l\left(\frac{q-N}{N-1}+l\right)!C_l^j(-1)^{l-j}=(q-N)!\mu(j,q,N)\left[\prod_{s=0}^{N-1}\left(\frac{q-N}{N-1}+j+1+s\right)\right]^{-1}$				
$\mu(j,q,N)$ (equation (7)):	$N=2$	$\mu=q-2-j$	$q=2N-1$	$\mu=C_N^{j+2}\ldots$
$G=a\log\Delta+\frac{1}{\Delta}$	$l_{q-N}=1+\sum_{1}^{N-1}\eta_{j-1}\partial_t^j$	$\nu=a^{1-N}(-1)^{N+1}(N-1)!$	$q=2N-1$	
$\bar\eta_l$ (equation (6))	$\sum l!\bar\eta_l(-1)^l=0$	$\bar\eta_{N-2}=\frac{\nu}{(N-1)!}\left(\prod_{s=0}^{N-1}(N+s)\right)^{-1}$		
$\sum_{l=M-1}^{N-2} l!(-1)^l[(a+1)\bar\eta_l C_l^{M-1}-\bar\eta_{l-1}C_{l-1}^{M-2}]=\frac{(-1)^N\nu a^{N-M}C_N^M}{\prod_{s=0}^{N-1}(M+s)}$	$2\leqslant M\leqslant N-1$			
$N=3$	$q=5$	$G=\frac{1}{7}\log\Delta+\frac{1}{\Delta}$	$l_2=1+\frac{49}{60}\partial_t(1+\partial_t)$	$\nu=98$
$N=4$	$q=7$	$G=a\log\Delta+\frac{1}{\Delta}$	$l_3=1-\frac{6}{a^3 7!}\partial_t[5-27a+(6-27a)\partial_t+\partial_t^2]$	
$\nu=-6a^{-3}$	$75a^2+2-17a=0$			

and we apply a direct factorisation method building simultaneously l_{q-N} and $X(\Delta^{-1})$. Different ways will be open depending whether bp is a positive integer, k, or not. We must kill all Δ^{-m} terms having $m<Np$, this will define a first operator $\mathscr{L}_{\rm I}$ and the complementary operator $\mathscr{L}_{\rm II}$ will reconstruct exactly the polynomial $X(\Delta^{-1})$. We introduce a first order ∂_t operator raising by one unit power a sum of two successive Δ^{-m} terms

$$(1+m^{-1}\partial_t)\Delta^{-m}(A+B\Delta^{-1})=\Delta^{-(m+1)}(A-m^{-1}B)+m^{-1}(m+1)B\Delta^{-(m+2)} \tag{8}$$

and apply it k times to G

$$\prod_{s=p}^{p+k-1}(1+s^{-1}\partial_t)G=(b-kp^{-1})\Delta^{-(p+k)}+(1+kp^{-1})\Delta^{-(p+k+1)}.$$

As long as $p+k<Np$, two possibilities occur: either $bp=k$ or not. In the $bp=k$ case, we have only one Δ^{-Np} term and $\mathscr{L}_{\rm II}$ is an operator of order N. In the $bp\neq k$ case, we have two terms Δ^{-Np} and $\Delta^{-(Np+1)}$ and $\mathscr{L}_{\rm II}$ is an $(N-1)$th-order operator. In both cases we obtain $q=(N-1)p+2N-1$ and $l_{q-n}=\mathscr{L}_{\rm I}\mathscr{L}_{\rm II}$.

(i) We assume $pb=k$ and further $p=(k+1)(N-1)^{-1}=$ integer $(N-1)^{-1}$. We find for $\mathscr{L}_{\rm I}$, $\mathscr{L}_{\rm II}$

$$\mathscr{L}_{\rm I}=\prod_{s=p}^{Np-2}(1+s^{-1}\partial_t)\qquad \mathscr{L}_{\rm I}G=(b+1)\Delta^{-Np}\qquad b=(N-1)(q-2N)(q-2N+1)^{-1}$$

$$\mathscr{L}_{\rm II}=1+\sum_{1}^{N}\eta_{j-1}\partial_t^j\qquad \mathscr{L}_{\rm II}(b+1)\Delta^{-Np}=X(\Delta^{-1})\qquad q\geqslant 2N+1. \tag{9a}$$

If $pb = k$ but $p+k+1 < Np$ we must use another differential operator $(1+B^{-1}\partial_t)A\Delta^{-B} = A\Delta^{-(B+1)}$ in order to go to Δ^{-Np}. In this case we find

$$\mathscr{L}_{\mathrm{I}} = \prod_{r=p+k+1}^{Np-1} (1+r^{-1}\partial_t) \prod_{s=p}^{p+k-1} (1+s^{-1}\partial_t),$$

$$\mathscr{L}_{\mathrm{I}}G = \frac{b+1}{\Delta^{Np}}, b = \frac{k(N-1)}{q+1-2N} \qquad q \geqslant 2N+2 \tag{9a'}$$

whereas $\mathscr{L}_{\mathrm{II}}$ is the same as the preceding one given in equation (9a). Both cases are

Table 3. (a) Solutions with two Δ and $b = k/p$. (b) Solutions with two Δ and $b \neq k/p$.

(a)

$G = \Delta^{-p}(b+\Delta^{-1})$ $\qquad p = \frac{q-N}{N-1} - 1$ $\qquad b = \frac{k}{p}$ k integer X (equation (4′)) $\quad l_{q-N}G = X$ $\quad \mathscr{L}_{\mathrm{II}}$ (equation (9a))

$b = \frac{(N-1)(q-2N)}{q+1-2N}$ $\mathscr{L}_{\mathrm{I}}$ (equation (9a)) $q \geqslant 2N+1$ $\qquad b = \frac{k(N-1)}{q+1-2N}$ $1 \leqslant k \leqslant q-2N-1$ $\mathscr{L}_{\mathrm{I}}$ (equation 9′a))

$q \geqslant 2N+2$

$N = 3$ $\qquad \frac{\eta_2}{\nu} = \frac{16(q-3)^2}{9(b+1)(q-5)^2(3q-13)(3q-11)(3q-7)(3q-5)}$

$$\frac{\eta_1}{\nu} = \frac{3}{2}\frac{\eta_2}{\nu}(3q-13) + \frac{4}{3}\frac{(q-3)b}{(b+1)(3q-13)(3q-11)(3q-7)}$$

$$\frac{\eta_0}{\nu} = \frac{\eta_1}{2\nu}(6q-28) - \frac{\eta_2}{4\nu}(4+9(q-5)(3q-13)) + \frac{2}{3}\frac{b^2(q-5)}{(3q-13)(3q-11)(b+1)}$$

$$\nu^{-1} = 3\left(\frac{q-5}{2}\right)\eta_0 - 9\left(\frac{q-5}{2}\right)^2\eta_1 + 27\left(\frac{q-5}{2}\right)^3\eta_2 + \frac{b^3(q-5)^2}{3(b+1)(3q-13)(3q-11)}$$

$N = 3$ $q = 7$ $G = \Delta^{-1} + \Delta^{-2}$ $\qquad l_4 = (1+\partial_t)\left(1 + \frac{\partial_t}{31}\left(\frac{173}{12} + 2\partial_t + \frac{1}{12}\partial_t{}^2\right)\right)\tilde{l}_3 G = \tilde{a}_{30}\frac{420}{31}(G_x)^3$

(b)

$G = \Delta^{-p}(b+\Delta^{-1})$ $\qquad p = \frac{q-N}{N-1} - 1$ $\qquad b \neq kp^{-1}$ $\qquad l_{q-N}$ (equation (9b))

$N = 3$ $\qquad p = \frac{q-5}{2}$ $\qquad L_q = \tilde{l}_3 l_{q-3}$ $\qquad l_{q-3} = (1+\eta_0\partial_t + \eta_1\partial_t^2)\prod_{s=(q-5)/2}^{3q-17/2}(1+s^{-1}\partial_t)$

$$b^3p^2(p+1)(4p+7) + 6pb^2(2p^3+6p^2+4p-1) + b(p+1)(12p^3+27p^2+8p-1) + (p+1)^3(4p+2) = 0$$

$q = 6$ $\qquad 9b^3 + 22b^2 + 45b + 36 = 0$ $\qquad 2 + \eta_1(-11b^2 - 39b - \frac{57}{2}) = 0$

$\eta_0 = \eta_1(7+6b)$ $\qquad \nu = \frac{5\times7\times11\times13}{4}\eta_1$

$q = 7$ $\qquad (b+1)(11b^2+22b+24) = 0$ $\qquad -1 + \eta_1(31b^2+62b+36) = 0$

$\eta_0 = 11\eta_1(b+1)$ $\qquad \nu = \eta_1 3\times4\times5\times6\times7$

if $b = -1$ $\qquad l_4 = (1+\frac{1}{5}\partial_t^2)(1+\frac{1}{2}\partial_t)(1+\partial_t)$ $\qquad \nu = 3\times4\times6\times7$ $\qquad G = -\frac{1}{\Delta} + \frac{1}{\Delta^2}$

given in table 3(a) and further for $N=3$ we give the explicit relations between η_i, ν, b, q which must be solved in the following order: b is known $\rightarrow \eta_i \nu^{-1}$ are functions of $q, b \rightarrow \nu(q, b) \rightarrow \eta_i(q, b)$. As an illustration for $N=3$, $q=7$, $k=b=p=1$, we explicitly write down the result.

(ii) We assume $b \neq kp$ as long as $p+k \leq Np$ and we find

$$\mathscr{L}_{\mathrm{I}} = \prod_{s=p}^{Np-1} (1+s^{-1}\partial_t) \qquad \mathscr{L}_{\mathrm{I}} G = \Delta^{-Np}(b-N+1+N\Delta^{-1}) l_{q-N} = \mathscr{L}_{\mathrm{I}}\mathscr{L}_{\mathrm{II}}$$

$$\mathscr{L}_{\mathrm{II}} = 1 + \sum_{1}^{N-1} \eta_{j-1}\partial_{tj} \qquad \mathscr{L}_{\mathrm{II}}\Delta^{-Np}(b-N+1+N\Delta^{-1}) = X(\Delta^{-1}). \tag{9b}$$

The main difference with the previous case is that b is unknown. From $\eta_i \equiv \eta_i(p, b)$, $\nu(p, b)$ one must find the algebraic equation for b, with p dependent coefficients, and then go back to the determination of η_i, ν. For $N=3$ (table 3(b)) we write down the cubic equation for b and for $q=6, 7$ we quote all the relations. We notice that for $q=7$, $b=-1$ is a simple solution for which we can give all the numerical values of l_4.

Of course one can go on and consider more than two Δ as was done in the $N=2$ case (Cornille and Gervois 1982a, b, c). Let us emphasise once more that power-like nonlinearities K^N and $K^{N-1}K_x$ share common features when the associated linear operators are factorised and the bi-solitons do not have soliton couplings. They consistute a class of non-completely integrable equations and in a separate publication we enlarge this class, by including $(\Sigma\lambda_i\partial_x^{i_i})K^N$ and $\Sigma\lambda_i K^{N_i}$ nonlinearities. Another interesting feature here is the fact that the generalisation of the Burgers's germ $\tilde{l}_2\Delta=0$ is obtained with simple linear differential relations $\tilde{l}_N\Delta^i=0$, $i=1,\ldots,N-1$. This result will be the starting point for the investigation of solutions associated to non-linearities $K^{N-1}K_x$ and different from the ones considered here.

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